Solution Sheet 6

Exercise 1. Let $P \in \mathbb{C}[X, Y, Z]$ be a non-zero irreducible, homogeneous polynomial. Let $C_P \subset \mathbb{P}^2_{\mathbb{C}}$ be the associated non-singular projective curve. Use the hints to show that C_P is connected for the Euclidean topology.
Hint 1: The Riemann-Roch theorem guarantees for any $x_0 \in C_P$ the existence of a meromorphic function on C_P with a pole only at x_0 of order $\leq r$ for r large enough, and holomorphic elsewhere.
Hint 2: A nonzero meromorphic function on an irreducible algebraic curve cannot vanish on a nonempty Euclidean open set. (*) If you know algebraic geometry, why?

Remark. The statement of Exercise 1 also holds if C_P has singular points.

Solution 1. Assume $C_P = M_1 \sqcup M_2$ with M_1, M_2 nonempty, disjoint, closed. Then both are also open since $M_2 = C_P \setminus M_1$. Pick $x_0 \in M_1$. By Riemann-Roch, for $r \gg 0$ there exists a nonconstant meromorphic function f on C whose only pole is at x_0. Thus f restricts to a holomorphic function on the open set M_2. Since C_P is projective, it is compact and hence the closed subset M_2 is compact. Therefore $|f|$ attains its maximum at some $m \in M_2$. Choose a disk $D \subset M_2$ around m. The maximum modulus principle implies that $f|_D \equiv \alpha$ for some $\alpha \in \mathbb{C}$. Consider the meromorphic function $f - \alpha$ on C_P. It is not identically zero (since f has a pole at x_0) but has infinitely many zeros on C_P (it vanishes on the nonempty open set D). This contradicts the basic fact that on an irreducible algebraic curve a nonzero meromorphic function cannot vanish on a nonempty Euclidean open set.

(*) Let C be an irreducible algebraic curve and η its unique generic point. Then the function field $\mathbb{C}(C)$ is identified with $\mathcal{O}_{C,\eta}$, the stalk of the structure sheaf $\mathcal{O}_C$ at η. For any $f \in \mathbb{C}(C)$ there exists a nonempty Zariski open set $U \subset C$ such that f is represented by a regular section $s \in \mathcal{O}_C(U)$. Suppose s vanishes on a nonempty Zariski open subset $W \subset U$. Then $\eta \in W$ and the germ of s at η is zero in $\mathcal{O}_{C,\eta}$. This forces $f = 0$ in $\mathbb{C}(C)$, a contradiction unless $f = 0$. Thus a nonzero meromorphic function cannot vanish on a nonempty Zariski open subset. Over $\mathbb{C}$, any nonempty Euclidean open subset in C is Zariski dense, so vanishing there forces vanishing on such a W.

Exercise 2. Let X, Y be compact connected Riemann surfaces and let $f : X \to Y$ be a nonconstant holomorphic map. Show that $g(X) \geq g(Y)$.

Solution 2. If $g(Y) = 0$, the statement is automatic because $g(X) \geq 0$ for every compact Riemann surface. For $x \in X$ we denote by e_x the ramification multiplicity of f at x. Since $e_p - 1 \geq 0$ for all p, the Riemann-Hurwitz formula gives

$$2g(X) - 2 \geq \deg(f)\big(2g(Y) - 2\big).$$

If $g(Y) = 1$, then this forces $g(X) \geq 1$. If $g(Y) \geq 2$, then $2g(X) - 2 \geq 2g(Y) - 2$, that is, $g(X) \geq g(Y)$.

Exercise 3. (for credit, due on 2 November)
Let X be a compact connected Riemann surface of genus g and let $f : X \to \mathbb{P}^1$ be a nonconstant holomorphic map. Let r be the number of branch values of f. For $x \in X$ we denote by e_x the ramification multiplicity of f at x.

1. *(2 points)* Let d be the degree of f. Show that $\sum_{x \in X}(e_x - 1) \leq r(d - 1)$.
2. *(1.5 points)* If $r \leq 1$, show that f is a biholomorphism.
3. *(1.5 points)* If $r = 2$, show that $g = 0$.
4. Explore the case $r = 3$ (Belyi functions, dessin d'enfant).

Solution 3. (1) Set $R = \sum_{x \in X}(e_x - 1)$. If y is not a branch value then the contribution of the fiber $f^{-1}(y)$ to R is 0. If y is a branch value, then we have

$$\sum_{x \in f^{-1}(y)} (e_x - 1) = d - \# f^{-1}(y) \leq d - 1.$$

Summing over the r branch values gives

$$R = \sum_{x \in X} (e_x - 1) \leq r(d - 1).$$

(2) If $r \leq 1$, then by part (1) we have $R \leq d - 1$. Plugging this into the Riemann-Hurwitz formula yields

$$2g - 2 = d\big(2g(\mathbb{P}^1) - 2\big) + R = -2d + R \leq -(d + 1).$$

Since always $2g - 2 \geq -2$, we must have $d \leq 1$. As f is nonconstant, $d = 1$. This means that f is a biholomorphism. Note that this means there is no branching at all. So there is no non-constant map $X \to \mathbb{P}^1$ with exactly one branch point.

(3) If $r = 2$, then by part (1) we have $R \leq 2(d - 1)$. From the Riemann-Hurwitz formula we get

$$2g - 2 = -2d + R \leq -2d + 2(d - 1) = -2.$$

This implies $g \leq 0$ and therefore $g = 0$.

Exercise 4. Let X be a compact connected Riemann surface, G a finite group acting holomorphically and properly on X, and $\pi : X \to Y := X/G$ the quotient map. Suppose there are exactly r orbits of points with non-trivial stabilizer; choose representatives $p_1, \ldots, p_r$ with stabilizer orders $m_j > 1$. Prove the formula

$$2g(X) - 2 = |G|\big(2g(Y) - 2\big) + |G| \sum_{j=1}^{r} \Big(1 - \frac{1}{m_j}\Big). \tag{1}$$

Solution 4. Riemann-Hurwitz for π says

$$2g(X) - 2 = \deg(\pi)\,\big(2g(Y) - 2\big) + \sum_{x \in X} \big(e_x - 1\big).$$

We have $\deg(\pi) = |G|$. Points with trivial stabilizer contribute $e_x - 1 = 0$ to the sum. For an orbit with stabilizer order m_j, every point in the orbit has ramification multiplicity $e_x = m_j$, and the orbit size is $|G|/m_j$, so its total contribution to the sum is

$$\frac{|G|}{m_j}(m_j - 1) = |G|\Big(1 - \frac{1}{m_j}\Big).$$

Summing over the r nonfree orbits yields the formula.

Exercise 5. Let G be a finite group acting by automorphisms on $\mathbb{P}^1$ and let $\pi : \mathbb{P}^1 \to Y := \mathbb{P}^1/G$ be the quotient map. Show that π cannot be ramified over exactly one point of Y.

Solution 5. Note that the case $|G| = 1$ has no branching, so we assume $|G| \geq 2$. Assume π has exactly one branch value whose preimage consists of points each having stabilizer order $m > 1$. The Riemann-Hurwitz formula from the previous exercise gives

$$2 - 2g(Y) = \frac{2}{|G|} + \Big(1 - \frac{1}{m}\Big).$$

Since the right hand side is positive, $g(Y) = 0$. This gives a contradiction. Hence π cannot have only one branch value.

Exercise 6.
Fix an integer $m \geq 2$. Let

$$C = \{[X : Y : Z] \in \mathbb{P}^2 : X^m + Y^m = Z^m\}$$

be the projective Fermat curve of degree m. Compute the genus of C.
Hint: Let $\mu_m = \{\zeta \in \mathbb{C}^\times : \zeta^m = 1\}$ act on C by

$$\zeta \cdot [X : Y : Z] = [X : \zeta Y : Z].$$

Show that C/μ_m is biholomorphic to $\mathbb{P}^1$ and apply the Riemann-Hurwitz formula from Exercise 4.

Solution 6. To identify the quotient, consider

$$\phi : C \to \mathbb{P}^1, \quad [X : Y : Z] \mapsto [X : Z].$$

This map is invariant under the action $\zeta \cdot [X : Y : Z] = [X : \zeta Y : Z]$. One can check that for any $[x : z] \in \mathbb{P}^1$, the fiber $\phi^{-1}([x : z])$ consists of exactly one μ_m-orbit. The quotient map $\pi : C \to C/\mu_m$ identifies each orbit to a single point, so the induces map $\overline{\phi} : C/\mu_m \to \mathbb{P}^1$ is bijective, and hence a biholomorphism.
Next, we identify points $[X : Y : Z] \in C$ with nontrivial stabilizer. A point $[X : Y : Z] \in C$ has a nontrivial stabilizer if and only if there exists a m-th root of unity $\zeta \neq 1$ with $(X, Y, Z) = \lambda(X, \zeta Y, Z)$ for some $\lambda \neq 0$.

- In the affine chart $Z = 1$, this means that $(x, y, 1) = \lambda(x, \zeta y, 1)$, from which we deduce $\lambda = 1$. Thus in this chart the only nontrivial fixed points satisfy $y = 0$. From the curve equation we get $x^m = 1$, yielding the m points

  $$(\zeta^k, 0), \quad (k = 0, \ldots, m-1).$$

  Each of these is fixed by the whole group μ_m.
- At infinity $Z = 0$, the fixed point condition is $(X, Y, 0) = \lambda(X, \zeta Y, 0)$, from which we deduce $\lambda = 1$. Then $Y = \zeta Y$ with $Y \neq 0$ forces $\zeta = 1$. Thus there is no nontrivial stabilizer at infinity.

The two cases $Z \neq 0$ and $Z = 0$ cover all points of C, so the classification is complete.
Now we apply the Riemann-Hurwitz formula to the quotient map $\pi : C \to C/\mu_m$. With $|G| = m$, $g(Y) = 0$, $r = m$, and $m_j = m$ for all j, we deduce from (1) that

$$g(C) = \frac{(m-1)(m-2)}{2}.$$